# Nonclassicality and the concept of local constraints on the photon number distribution

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We exploit results from the classical Stieltjes moment problem to bring out the totality of all the information regarding phase insensitive nonclassicality of a state as captured by the photon number distribution $p_n$. Central to our approach is the realization that $n!\,p_n$ constitutes the sequence of moments of a (quasi) probability distribution, notwithstanding the fact that $p_n$ can by itself be regarded as a probability distribution. This leads to classicality restrictions on $p_n$ that are local in $n$ involving $p_n$'s for only a small number of consecutive $n$'s, enabling a critical examination of the conjecture that oscillation in $p_n$ is a signature of nonclassicality.

Nonclassical states of the radiation field continue to receive much attention. These are states for which the $P$-distribution $\varphi(z)$ is not a true probability. Prominent among the quantitative characteristics of nonclassicality are squeezing and sub-Poissonian statistics. While these involve the lower order moments of $\varphi(z)$, there have also emerged criteria involving the higher order moments. Among these we may note the higher order squeezing criteria of Hong and Mandel [3], the related amplitude squared squeezing introduced by Hillery [4], and the generalization of the Mandel Q-parameter achieved by Agarwal and Tara [5].

There has also emerged a qualitatively different kind of criterion for nonclassicality. While $p_n = \langle n|\hat{\rho}|n\rangle$, which represents the probability of there being $n$ photons in the state specified by the density operator $\hat{\rho}$, is a smooth function of $n$ for classical states like the coherent states and the thermal states, it is an oscillating function of $n$ for nonclassical states like the squeezed states, as was exposed in the seminal work of Schleich and Wheeler [6] on interference in phase space. Oscillation in $p_n$ has since then been taken as a signature of nonclassicality [7,8]. Indeed, these oscillations have come to be known as *nonclassical oscillations* [9]. It should, however, be noted that this oscillation criterion for nonclassicality, though insightful, has not been derived from basic principles and hence enjoys only the status of a conjecture. Its principal virtue lies in the fact that it is *local* in $n$, in contradistinction to the criteria involving the Mandel Q-parameter or its generalizations; the latter are expressed in terms of the moments of $p_n$, and hence are *global* in $n$.

For a radiation mode described by annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ measurements of operators which are functions of $\hat{a}^\dagger\hat{a}$ (the so called phase insensitive operators) do not depend on all the details of $\varphi(z)$, but are fully determined by the angle averaged radial "marginal" distribution $\mathcal{P}(I)$, derived from $\varphi(z)$ by writing $z = I^{1/2}e^{i\theta}$ and averaging over $\theta$:

$$\mathcal{P}(I) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \varphi(I^{1/2}e^{i\theta})\,. \qquad (1)$$

In particular we have

$$p_n \;=\; \int_0^\infty dI\tilde{\mathcal{P}}(I)\frac{I^n}{n!}\,, \qquad \tilde{\mathcal{P}}(I) = \mathcal{P}(I)e^{-I}\,, \qquad (2)$$

where $n = 0, 1, 2, \ldots$ The above relation is invertible. That is, the sequence $\{p_n\}$ represents $\mathcal{P}(I)$ faithfully.

For a given state, it may happen that $\varphi(z)$ is not a true probability, but the phase averaged $\mathcal{P}(I)$ is a bonafide probability distribution. Such states (the Yurke-Stoler state [10] is an example) are said to exhibit phase sensitive nonclassicality. On the other hand the nonclassicality of the state may be such that it survives the process of phase averaging involved in (1), thereby rendering $\mathcal{P}(I)$ itself a quasiprobability rather than a true probability. Then we talk of (the stronger) phase insensitive nonclassicality. Clearly, any state with sub-Poissonian statistics is nonclassical of the phase insensitive type.

The purpose of this Letter is to exhibit the totality of all the information regarding nonclassicality of a state as captured by the photon number distribution sequence

$\{p_n\}$ or, equivalently, by $\mathcal{P}(I)$. Since we work at the level of $\{p_n\}$, and not $\varphi(z)$, only states with phase insensitive nonclassicality will be said to be "nonclassical". All other states will be termed as "classical", for brevity.

*The key to our approach is an appreciation of the fact that* $\{p_n\}$ *is essentially the moment sequence of a (quasi) probability distribution, notwithstanding the fact that* $p_n \geq 0$ *and* $\sum p_n = 1$*, and hence* $\{p_n\}$ *can legitimately be viewed as a probability distribution over the discrete variable* $n$. This departure from tradition leads us to derive constraints on $p_n$ which are local in $n$ involving $p_n$'s for only a small number of consecutive $n$'s, and enables us to critically examine the oscillation criterion in a direct manner. Necessary and sufficient conditions for absence of (phase insensitive) nonclassicality in a state are presented, not only in terms of the sequence $\{p_n\}$ but also in the (dual) traditional approach involving the factorial moments of $\{p_n\}$.

*Local constraints on classical* $\{p_n\}$.— It turns out to be convenient to define a sequence $\{q_n\}$ in the place of $\{p_n\}$ through $q_n = n! p_n$, for $n = 0, 1, 2, \ldots$ It follows from (2) that $\{q_n\}$ is simply the moment sequence of the distribution $\tilde{\mathcal{P}}(I) = \mathcal{P}(I)e^{-I}$:

$$q_n \;=\; \int_0^\infty dI \tilde{\mathcal{P}}(I) I^n \equiv < I^n >_{\tilde{\mathcal{P}}} \;. \tag{3}$$

Now suppose we are given a classical state so that $\tilde{\mathcal{P}}(I) \geq 0$, for $0 \leq I < \infty$, and consider the polynomial $f(I) = I^n (I - x)^2$. Since $f(I)$ is manifestly nonnegative for any real value of the parameter $x$, nonnegativity of $\tilde{\mathcal{P}}(I)$ implies, through (3),

$$\begin{aligned} < f(I) >_{\tilde{\mathcal{P}}} \; &= \langle x^2 I^n - 2xI^{n+1} + I^{n+2} \rangle_{\tilde{\mathcal{P}}} \\ &= x^2 q_n - 2x q_{n+1} + q_{n+2} \;\geq\; 0 \,, \end{aligned} \tag{4}$$

for all real $x$. That is, $q_n \,, q_{n+2} \geq 0$ and

$$q_n q_{n+2} \;\geq\; q_{n+1}^2 \,, \qquad n = 0, 1, 2, \ldots \tag{5}$$

Written in terms of $\{p_n\}$, the above condition reads

$$p_n p_{n+2} \;\geq\; (\frac{n+1}{n+2}) p_{n+1}^2 \;. \tag{6}$$

These are our *local conditions* to be necessarily satisfied by the photon distribution $\{p_n\}$ of any classical state.

Several interesting conclusions can be drawn from these conditions which are local in $n$, and are saturated for every value of $n$ by any Poissonian distribution. Suppose that we are given a state for which $p_{n_0} = 0$ (and hence $q_{n_0} = 0$) for some integer $n_0 \geq 0$, and assume that the state is classical. The choice $n = n_0$ in the local condition (5) implies that $q_{n_0+1} = 0$. Similarly the choice $n+2 = n_0$ implies $q_{n_0-1} = 0$. Continuing this process we find that for a classical state either $p_n$ is nonzero for every values of $n$, or $p_n = 0$ for all $n > 0$. In other words, *a classical state other than the vacuum state, cannot be orthogonal to any Fock state.* To appreciate the significance of this conclusion, consider the state

$$\hat{\rho} = N \hat{a}^{\dagger m} \hat{\rho}_0 \hat{a}^m \;, \tag{7}$$

where $\hat{\rho}_0$ is an arbitrary density operator, and $N$ is a normalization constant. We can call it the "photon added" $\hat{\rho}_0$, for it includes the photon added coherent state [11] and the photon added thermal state [5,12] as special cases. Since $p_n =< n|\hat{\rho}|n >= 0$ for $n < m$, we conclude that $\hat{\rho}$ is nonclassical for every $m > 0$. Thus, we have established the following result: *all photon added states, pure or mixed, are nonclassical.*

There has been remarkable progress recently in quantum state reconstruction using techniques of optical homodyne tomography [13]. Thus, it is now possible to 'map out' the Wigner distribution of a state using the inverse Radon transform, or reconstruct the density matrix in the Fock basis using a set of pattern functions. Schiller *et al* [14] report such a reconstructed $\hat{\rho}$ upto $n = 6$, with $q_n = 0.44$, 0.07, 0.26, 0.30, 1.44, 3.60, 28.80. The local conditions (5) are violated (for instance, $0.07 \times 0.30 \leq (0.26)^2$). Thus, the squeezed vacuum of Schiller *et al* turns out to be a nonclassical state of the phase insensitive type. That is, the nonclassicality of their state survives phase averaging, notwithstanding the fact that consideration of the Mandel $Q$-parameter will do no more than to simply indicate that this state is "strongly super-Poisonian" as noted by the authors.

We now turn to the oscillation criterion for nonclassicality. There exist classical states for which the photon distribution $\{p_n\}$ is an oscillatory function of $n$. We may call these *classical oscillations*. Figure 1 shows such classical oscillations for an incoherent mixture of suitably chosen coherent states:

$$\hat{\rho} = \sum \lambda_j \, |\alpha_j\rangle\langle\alpha_j| \;, \qquad \sum \lambda_j = 1 \,. \tag{8}$$

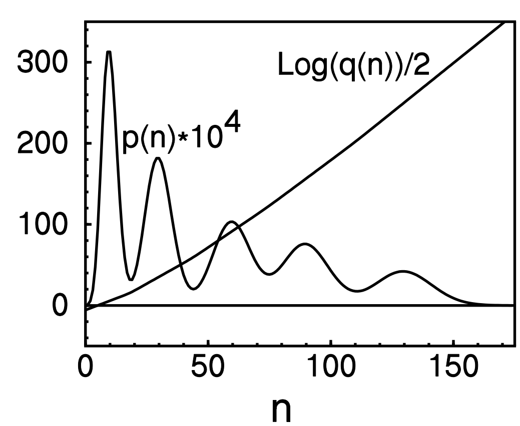

FIG. 1. Showing classical oscillations in the photon distribution for an incoherent mixture of five coherent states with $\lambda_j = 0.25$, 0.25, 0.2, 0.18, 0.12 and corresponding $|\alpha_j|^2 = 10$, 30, 60, 90, 130. Here $p(n)$ and $q(n)$ stand, respectively, for $p_n$ and $q_n$ of the text. Note that $q_n$ exhibits no oscillation for this classical state

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On the other hand the photon added thermal (or coherent) state has a $\{p_n\}$ with no oscillation; we have nevertheless seen that it is a nonclassical state. This, however, should not tempt one to simply dismiss the oscillation criterion as being neither sufficient nor necessary for nonclassicality; for, as already noted, the virtue of this criterion does not reside in its exactitude, but rather in its distinction of being local in $n$. Thus, it is highly desirable to amend it suitably but without sacrificing its local character. This is easily achieved through our local conditions. To see this, note that (5) implies that $\{q_n\}$ for a classical state cannot have a local maximum (if it had, the inequality will be violated by allowing $n+1$ to correspond to the local maximum), and hence cannot exhibit any oscillation. Thus we arrive at the desired modification: *oscillation in* $\{q_n\}$ *is a sufficient condition for nonclassicality*. It is $\{q_n\}$, and not $\{p_n\}$, that is the key to the correct oscillation criterion. Indeed, $\{p_n\}$ can oscillate for a classical state with amplitude limited by the extent permitted by the $(n+1)/(n+2)$ factor in (6). Even period-two classical oscillations are allowed, as can demonstrated using classical states of the type (8).

Finally, we apply our local conditions to a class of states obtained as superposition of two coherent states:

$$|\Psi >= N[|z_0 > +e^{i\theta}| - z_0 >], \tag{9}$$

where $\theta$ is the relative phase (in the Pancharatnam [15] sense) between the two components of the superposition, and $N$ is the normalization constant. We have

$$\frac{q_n q_{n+2}}{q_{n+1}^2} = \frac{(1+(-1)^n \cos\theta)^2}{(1+(-1)^{n+1}\cos\theta)^2}\,. \tag{10}$$

It is clear that our local conditions (5) are violated by $|\Psi\rangle$ for all values of $\theta \neq \pm\pi/2$; by odd values of $n$ for $-\pi/2 < \theta < \pi/2$, and by even values of $n$ for the range $-3\pi/2 < \theta < -\pi/2$. It is well known [16] that coherent states are the only pure states for which the $\mathcal{P}$-distribution $\varphi(z)$ is a true probability. Further, the Yurke-Stoler [10] states, which correspond to $\theta = \pm\pi/2$, have Poissonian $\{p_n\}$ and hence possess only phase sensitive nonclassicality. Thus, what is striking about the above analysis is the inference that for all values of $\theta \neq \pm\pi/2$ the superposition state has *phase insensitive* nonclassicality, and that it is exposed by our lowest order local conditions!

*Necessary and sufficient condition for nonclassicality.*— We showed that positivity of $\tilde{\mathcal{P}}(I)$ implies the local conditions (5) on its moment sequence $\{q_n\}$. We now exploit results from the classical problem of moments to exhibit the necessary and sufficient conditions on $\{q_n\}$ in order that the associated state $\hat{\rho}$ is classical.

The classical moment problem, on which there exists an enormous amount of literature [17], consists of two parts: (i) to test if a given sequence of numbers qualifies to be the sequence of moments of some bonafide probability distribution, and (ii) to reconstruct a probability distribution from its moment sequence. If the probability distribution is over the semi-infinite real line $[0,\infty)$, one calls it the Stieltjes moment problem. The Hamburger moment problem corresponds to the case where the probability distribution is over the entire real line $(-\infty,\infty)$. Since, $I = |z|^2 \geq 0$, our problem of deriving the necessary and sufficient condition on the moment sequence $\{q_n\}$ in order that $\tilde{\mathcal{P}}(I)$ is a true probability distribution is indeed a Stieltjes moment problem.

Solution of this classical problem is well known [17]. To exhibit this solution, construct from the moment sequence $\{q_n\}$ two $(N+1)$-dimensional symmetric square matrices $L^{(N)}$ and $\tilde{L}^{(N)}$ as follows:

$$L^{(N)}_{mn} = \begin{pmatrix} q_0 & q_1 & q_2 & \cdots & q_N \\ q_1 & q_2 & q_3 & \cdots & q_{N+1} \\ \vdots & \vdots & \vdots & & \vdots \\ q_N & q_{N+1} & q_{N+2} & \cdots & q_{2N} \end{pmatrix},$$

$$\tilde{L}^{(N)}_{mn} = \begin{pmatrix} q_1 & q_2 & q_3 & \cdots & q_{N+1} \\ q_2 & q_3 & q_4 & \cdots & q_{N+2} \\ \vdots & \vdots & \vdots & & \vdots \\ q_{N+1} & q_{N+2} & q_{N+3} & \cdots & q_{2N+1} \end{pmatrix}. \tag{11}$$

*Theorem 1:* The necessary and sufficient condition on the photon number distribution sequence $\{q_n = n!p_n\}$ of a state $\hat{\rho}$, in order that the associated quasiprobability distribution $\tilde{\mathcal{P}}(I)$ is a true probability, is that the matrices $L^{(N)}$, $\tilde{L}^{(N)}$ be nonnegative:

$$L^{(N)} \geq 0\,, \quad \tilde{L}^{(N)} \geq 0\,, \quad N = 0,1,2,\ldots \tag{12}$$

It may be noted in passing that for the Hamburger moment problem on the entire real line $(-\infty,\infty)$, the condition $L^{(N)} \geq 0$ is both necessary and sufficient.

It is immediate to relate our local condition to the above theorem. Nonnegativity of $L^{(N)}, \tilde{L}^{(N)}$ demands, as a necessary condition, nonnegativity of the diagonal $2\times 2$ blocks of $L^{(N)}, \tilde{L}^{(N)}$. This is precisely what our local conditions (5) are! It is also clear why our local conditions (5) are not sufficient: positivity of the diagonal $2\times 2$ blocks of $L^{(N)}$, $\tilde{L}^{(N)}$ does not capture in its entirety the positivity of $L^{(N)}$ and $\tilde{L}^{(N)}$ required in (12).

We can derive the next higher level of local conditions for classicality using our necessary and sufficient conditions (12). Given the sequence $\{q_n\}$, we define

$$x_n = q_n q_{n+2}/q_{n+1}^2\,, \qquad n = 0,1,2,\ldots \tag{13}$$

Then our first order local conditions (5) involving $q_n$ for three successive values of $n$ simply reads $x_n \geq 1$, $\forall n$, for any classical state. The second order local condition to be presented involves $q_n$ for five successive values of $n$ or, equivalently, $x_n$ for three successive values of $n$.

A necessary condition for the nonnegativity of $L^{(N)}$, $\tilde{L}^{(N)}$ is that their diagonal $3 \times 3$ blocks (such a block involves $q_n$ for five successive values of $n$) be nonnegative. After some algebra this condition can be written in terms of the $x_n$'s as

$$(x_n - 1)(x_{n+2} - 1) \geq (\frac{x_{n+1} - 1}{x_{n+1}})^2, \tag{14}$$

for $n = 0, 1, 2, \ldots$ These are our *second order local conditions* on $\{q_n\}$ or, equivalently, on $\{p_n\}$. They involve three successive $x_n' s$ and hence five successive $p_n' s$. Just like the first order conditions, these too are only necessary conditions for classicality, and we can similarly derive successive higher levels of local conditions.

To see an interesting implication of (14), recall that a Poissonian distribution $\{p_n\}$, *i.e.* a geometric sequence $\{q_n\}$, saturates the first order local conditions and renders $x_n = 1$ identically. We now ask whether it is possible to have a classical state for which $q_n q_{n+2} = q_{n+1}^2$ for some values of $n$, whereas $q_n q_{n+2} > q_{n+1}^2$ for other values of $n$. Such classical states, if they exist, can be said to be *locally Poissonian* at these former values of $n$.

Suppose a classical state is locally Poissonian at some $n = n_0$. That is, $x_{n_0} = 1$. Then two applications of (14), once with $n_0 = n$ and then with $n_0 = n + 2$, shows that the state will cease to be classical unless $x_{n_0+1} = 1$ and $x_{n_0-1} = 1$. Continuing this process we find that $x_n = 1$ for all $n$. Thus, there exists no non-Poissonian classical state which is locally Poissonian: *A classical state is either everywhere locally Poissonian* ($x_n = 1$ *for all* $n$) *or is everywhere locally super-Poissonian* ($x_n > 1$ *for all* $n$).

In the light of this result we can now strengthen and refine our first order condition (5) by adding that for a classical state these inequalities are either saturated for all $n$, or they are strict inequalities for all $n$.

*Factorial moments.*— We now present a dual approach to nonclassicality based on the traditional normal ordered moments $\gamma_n = \mathrm{tr}\left(\hat{a}^{\dagger n} \hat{a}^n \hat{\rho}\right)$. This approach will be seen to be along the lines of Agarwal and Tara [5]. However, the conditions for nonclassicality that we present are both *necessary* and *sufficient.*

Suppose we have a state $\hat{\rho}$ whose normal ordered moments $\gamma_n$ (*i.e.* factorial moments $\sum_k (k!)^{-1} (n+k)! p_{n+k}$ of $p_n$) are known. Our problem is to find necessary and sufficient conditions on the sequence $\{\gamma_n\}$ in order that the state $\hat{\rho}$ is classical. Transcribing $\gamma_n$ to the representation in terms of the $\mathcal{P}$-distribution $\varphi(z)$, and writing $z = I^{1/2} e^{i\theta}$, we have

$$\gamma_n = \int_0^\infty dI \mathcal{P}(I) I^n = \langle I^n \rangle_{\mathcal{P}} \,. \tag{15}$$

That is $\{\gamma_n\}$ is the moment sequence of $\mathcal{P}(I)$, in exactly the same manner in which the sequence $\{q_n\}$ was related to $\tilde{\mathcal{P}}(I)$. And the state $\hat{\rho}$ being classical is equivalent to $\mathcal{P}(I)$ being a true probability distribution. Thus, we have a Stieltjes moment problem once again, with solution parallel to the earlier one. Using the moment sequence $\{\gamma_n\}$, form $(N+1)$-dimensional symmetric matrices $M^{(N)}, \tilde{M}^{(N)}$ defined by

$$M_{jk}^{(N)} = \gamma_{j+k} \,, \qquad \tilde{M}_{jk}^{(N)} = \gamma_{j+k+1} \,. \tag{16}$$

where $j, k = 0, 1, \ldots, N$ and $N = 0, 1, 2, \ldots$

*Theorem 2:* The necessary and sufficient condition that the state $\hat{\rho}$ with normal ordered (*i.e.* factorial) moment sequence $\{\gamma_n\}$ be classical is that

$$M^{(N)} \geq 0 \,, \quad \tilde{M}^{(N)} \geq 0 \,, \quad N = 0, 1, 2, \ldots \tag{17}$$

It should be appreciated that theorem 2 completes the work initiated by Agarwal and Tara [5] by improving their necessary condition for classicality (they had only the condition $M^{(N)} \geq 0$) into the necessary and sufficient condition (17). Thus, the constraints on the factorial moments $\{\gamma_n\}$ arising from the requirement $M^{(N)} \geq 0$ are the same as in their work. However the additional constraints on these moments arising from the positivity requirement on $\tilde{M}^{(N)}$ are new: with $N = 0$ we have $\gamma_1 \geq 0$, with $N = 1$ we have $\gamma_1 \gamma_3 \geq \gamma_2^2$, and so on. It should be appreciated that these conditions cannot indeed be deduced from $M^{(N)} \geq 0$.

Considering diagonal $2 \times 2$ blocks of $M^{(N)}$, $\tilde{M}^{(N)}$ we obtain the classicality conditions $\gamma_k \gamma_{k+2} \geq \gamma_{k+1}^2$, for $k = 0, 1, 2, \ldots$ Clearly, these are dual to our first order local conditions (5). We may also derive conditions analogous to our second order local conditions (14), and so on.

To conclude, either of the two approaches based respectively on the moments of $\hat{\mathcal{P}}(I)$ and $\mathcal{P}(I)$ leads to complete solution of the problem of (phase insensitive) nonclassicality as coded in the photon number distribution $\{p_n\}$. It should however be appreciated that $q_n$'s are well defined for every state whereas the factorial moments $\gamma_n$ may not be finite, for $p_n$ may not decay fast enough as a function of $n$. For the states for which $\gamma_n$ exists for all $n$, the two approaches are equivalent. Even then, it is unlikely that the connection between nonclassicality and oscillation in $p_n$ could have been so easily settled in terms of $\gamma_n$. Finally, the first approach in terms of local conditions of $p_n$ has a distinct advantage at least in situations where, for some reason, $p_n$ is known not for all values of $n$. The density matrix from Ref. [14] which we analysed is such an example. The point is, even with knowledge of $p_n$ only for a finite set of values of $n$ one can now look for signatures of nonclassicality.

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